Chapter 3

FACTORIALS

A sequence which occurs frequently in mathematics is

$$1,\ 1,\ 1\cdot 2,\ 1\cdot 2\cdot 3,\ 1\cdot 2\cdot 3\cdot 4,\ \ldots .$$

We tabulate this in the form

n	0	1	2	3	4	5	6	...
n!	1	1	2	6	24	120	720	...

where the notation *n!* (read as **n factorial**) is used for the number on the second line that is thus associated with *n*. Clearly 0! = 1, 1! = 1, 2! = 2, 3! = 6, 4! = 24, etc. The definition of *n!* can be given as follows:

$$0! = 1,\ 1! = 1(0!),\ 2! = 2(1!),$$
$$3! = 3(2!),\ \ldots,\ (n+1)! = (n+1)(n!),\ \ldots .$$

The expression *n!* is not defined for negative integers *n*. One reason is that the relation $(n + 1)! = (n + 1)(n!)$ becomes $1 = 0\cdot(-1)!$ when $n = -1$, and hence there is no way to define (-1)! so that this relation is preserved.

Problems for Chapter 3

1. Find the following:

   (a) 7!.
   (b) $(3!)^2$.
   (c) $(3^2)!$.
   (d) (3!)!.

2. Find the following:

   (a) 8!.
   (b) (2!)(3!).
   (c) $(2\cdot 3)!$.

3. Show that $\binom{5}{2}(2!)(3!) = 5!$ and $\binom{7}{3}(3!)(4!) = 7!$.

4. Find c and d, given that $\binom{6}{2}(2!)(4!) = c!$ and $\binom{8}{3}(3!)(5!) = d!$.

5. Write as a single factorial:

   (a) $3! \cdot 4 \cdot 5$.
   (b) $4! \cdot 210$.
   (c) $n!(n + 1)$.

6. Express $a!(a^2 + 3a + 2)$ as a single factorial.

7. Find a and b such that $11 \cdot 12 \cdot 13 \cdot 14 = a!/b!$.

8. Find e, given that $(n + e)!/n! = n^3 + 6n^2 + 11n + 6$.

9. Express $(n + 4)!/n!$ as a polynomial in n.

10. Find numbers $a, b, c, d,$ and e such that $(n + 5)!/n! = n^5 + an^4 + bn^3 + cn^2 + dn + e$.

11. Calculate the following sums:

    (a) $1! \cdot 1 + 2! \cdot 2 + 3! \cdot 3$.
    (b) $1! \cdot 1 + 2! \cdot 2 + 3! \cdot 3 + 4! \cdot 4$.
    (c) $1! \cdot 1 + 2! \cdot 2 + 3! \cdot 3 + 4! \cdot 4 + 5! \cdot 5$.

12. Conjecture a compact expression for the sum $1! \cdot 1 + 2! \cdot 2 + 3! \cdot 3 + \cdots + n! \cdot n$ and test it for several values of n.

13. Show that $(n + 1)! - n! = n! \cdot n$.

14. Show that $(n + 2)! - n! = n!(n^2 + 3n + 1)$.

15. Find numbers $a, b,$ and c such that $(n + 3)! - n! = n!(n^3 + an^2 + bn + c)$ holds for $n = 0, 1, 2, \ldots$.

16. Use the formula in Problem 13 to derive a compact expression for the sum in Problem 12.

17. Use the formula in Problem 14 to derive a compact expression for

$$0! + 11(2!) + 29(4!) + \ldots + (4m^2 + 6m + 1)[(2m)!].$$

18. Derive a compact expression for

$$5(1!) + 19(3!) + 41(5!) + \ldots + (4m^2 + 2m - 1)[(2m - 1)!].$$

19. Derive compact expressions for:

(a) $0! + 5(1!) + 11(2!) + \ldots + (n^2 + 3n + 1)(n!)$.

*(b) $0! + 2(1!) + 5(2!) + \ldots + (n^2 + 1)(n!)$.

20. Derive a compact expression for $\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \ldots + \dfrac{n}{(n+1)!}$

21. Show that:

(a) $6! = 3! \cdot 2^3 \cdot 3 \cdot 5$.

(b) $8! = 4! \cdot 2^4 \cdot 3 \cdot 5 \cdot 7$.

(c) $10! = 5! \cdot 2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 9$.

22. Express r, s, and t in terms of m so that

$$1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2m - 1) = r!/(s! \cdot 2^t).$$